Greek Mathematical Olympiad – 2005

(February 12 / 2005)

1. We are given a trapeze ABCD with AB // CD , CD = 2 AB and DB $\perp$ BC. If lines DA and CB intersect at point E , prove that triangle CDE is isosceles (*modified*).

(Junior level)

2. If $f(n) = \dfrac{2n+1+\sqrt{n(n+1)}}{\sqrt{n+1}+\sqrt{n}}$ for all positive integers n , find the sum

$$A = f(1) + f(2) + \ldots + f(400)$$

(Junior level)

3. Let a circle and A an exterior point of the circle. Determine the points B, C, D on the circle , such that the convex quadrateral ABCD has the maximum area.

(Junior level)

4. Find the non zero integers a , b , c , d with a > b > c > d which are solutions of the system

$$\begin{cases} ab + cd = 34 \\ ac - bd = 19 \end{cases}$$

(Junior level)

5. Find all polynomials P(x) with real coefficients , P(2) = 12 and

$$P(x^2) = x^2 (x^2 + 1)P(x) \text{, for all real values of x.}$$

(Senior level)

6. Let the sequence (a_n) , $n \in \mathbb{N}^*$, with $a_1 = 1$ and

$$a_n = a_{n-1} + \frac{1}{n^3} \quad , \quad n = 2, 3, \ldots\ldots$$

a) Prove that $a_n < \frac{5}{4}$, for every n = 1 , 2 , …..

b) If ε is a positive real number , find the smallest natural $n_o > 0$, such that

$$|a_{n+1} - a_n| < \varepsilon \text{ , for all } n > n_0$$

(Senior level)

7. Let k a positive integer. If (x_o , y_o) is a solution of the equation

$$x^3 + y^3 - 2y(x^2 - xy + y^2) = k^2 (x - y) \quad (1)$$

with x_o , y_o non zero integers, prove that :

a) equation (1) has finite integer solutions (x , y) , with $x \neq y$

b) we can find 11 more different integer solutions (X, Y) of (1) with $X \neq Y$ where X, Y are functions of x_o and y_o

(Senior level)

8. Let xOy an (convex) angle and the rays Ox_1 , Oy_1 in it's interior , so that

$$\angle xOx_1 = \angle yOy_1 < \frac{1}{3} \angle xOy$$

Let K , L be fixed points on Ox_1 , Oy_1 respectively with OK = OL. If points A , B move on sides Ox , Oy respectively and the area of OAKLB is constant(has always the same value), prove that the circumcircle of triangle OAB , as A , B move on Ox , Oy , passes through a constant point , different from O.

(Senior level)